ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION TO THE SYSTEM OF EQUATIONS OF THERMAL COMBUSTION THEORY

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The propagation of a flame in a homogeneous gaseous mixture in which a first-order exothermic chemical reaction takes place is governed by the equations of mass diffusion and conservation of energy, which can be written in the form

$$\frac{d}{dx}\left(\frac{k}{c}\frac{du}{dx}\right) - m\frac{du}{dx} + v\Phi = 0,$$

$$\frac{d}{dx}\left(\rho D\frac{dv}{dx}\right) - m\frac{dv}{dx} - v\Phi = 0, \qquad (1)$$

$$-\infty \leqslant x \leqslant +\infty, \qquad u(-\infty) = u_-,$$

$$v(-\infty) = v_-, \qquad u(\infty) = u_+ . \qquad (2)$$

Here u is the temperature, $v = ha/c$, a is the concentration, h is the heat of reaction, c = const is the specific heat of the gas, $k = k(u)$ is the thermal conductivity, $\rho = \rho(u)$ is the gas density, $D = D(u)$ is the diffusion coefficient, m is the mass velocity, and $\Phi = \Phi(u)$ is the rate constant of the chemical reaction. The rate constant is $\Phi = 0$ for $u_- \leq u \leq \varepsilon$ and $\Phi > 0$ for $\varepsilon < u \leq u_+$.

The solution of the problem (1), (2) should determine the temperature and concentration profiles in the combustion wave and the speed of the wave m.

The problem of the existence and uniqueness of the solution of Eqs. (1), (2) was first considered in [1], in which it was proved that the problem always has a unique solution for $\lambda = \rho cD/k = 1$.

In [2] it was proved that the solution of (1), (2) exists for arbitrary non-zero constant values of λ and the uniqueness of the solution was proved for $0 < \lambda < 1$. The proof was based on the assumption that k, ρD, and c do not depend on temperature.

The existence and uniqueness of the solution of Eqs. (1), (2) in the case λ = 0 (D = 0), which corresponds to the propagation of the front of an exothermic reaction in a condensed medium, were investigated in [3].

Using Ya. B. Zel'dovich's method [1, 4], we shall prove that the problem (1), (2) also has a unique solution in the case that the thermal conductivity, the diffusion coefficient, and the density are functions of temperature such that $0 < \lambda(u) < 1$.

The system (1) has a first integral

$$\frac{k}{c}\frac{du}{dx} + \rho D\frac{dv}{dx} - m(u + v) = \mathrm{const}. \qquad (3)$$

From (3) and the conditions $(du/dx)_{x=\pm\infty} = (dv/dx)_{x=\pm\infty} = 0$, which follow from (2), we find

$$\rho D\frac{dv}{dx} = m(u + v - u_+) - \frac{k}{c}\frac{du}{dx}, \qquad u_+ = u_- + v_- . \qquad (4)$$

Taking the temperature u as an independent variable and introducing the new unknown function

$$p = \frac{k}{c}\frac{du}{dx} \qquad (5)$$

we obtain, instead of (1), (5),

$$\frac{dp}{du} = m - \frac{vf}{p} \qquad \left(f = \frac{k\Phi}{c}\right), \qquad (6)$$

$$\frac{dv}{du} = \frac{m(u + v - u_+)}{\lambda p} - \frac{1}{\lambda} \qquad \left(\lambda = \frac{\rho Dc}{k}\right), \qquad (7)$$

$$p(u_-) = p(u_+) = v(u_+) = 0 . \qquad (8)$$

The problem (6)–(8) is equivalent to the problem (1), (2). Over the interval $u_- \leq u \leq \varepsilon$, the solution of (6) with the condition $p(u_-) = 0$ is

$$p = m(u - u_-). \qquad (9)$$

As m increases from m = 0, the ordinates of the integral curves (9) also increase upwards from zero.

Consider the integral curves of Eqs. (6), (7) over the interval $\varepsilon \leq u \leq u_+$, which pass through the singular point $p(u_+) = 0$, $v(u_+) = 0$. Of the three integral curves passing through this point, only the curve with the slope

$$\left[\frac{dp}{du}\right]_{u_+} = \frac{m}{2\lambda(u_+)} - \left(\frac{m^2}{4\lambda^2(u_+)} + \frac{f(u_+)}{\lambda(u_+)}\right)^{1/2} \qquad (10)$$

is physically meaningful, and, consequently,

$$\left[\frac{dv}{du}\right]_{u_+} = \frac{2[1 - \lambda(u_+)]\, m}{\lambda(u_+)\,[m + (m^2 + 4\lambda(u_+) f(u_+))^{1/2}]} - 1 . \qquad (11)$$

The equations of the integral curves for m = 0 are

$$p(u, 0) = \left(2\int_u^{u_+} f(u_2)\left[\int_{u_2}^{u_+}\frac{du_1}{\lambda(u_1)}\right]du_2\right)^{1/2},$$

$$v(u, 0) = \int_u^{u_+}\frac{du_1}{\lambda(u_1)} . \qquad (12)$$

The solution of (7) can be written in the form

$$v(u, m) = u_+ - u +$$

$$+ \int_u^{u_+}\frac{1 - \lambda(u_2)}{\lambda(u_2)}\exp\left[-m\int_u^{u_2}\frac{du_1}{\lambda(u_1)\,p(u_1, m)}\right]du_2 . \qquad (13)$$

From (13) it follows that for $\lambda < 1$ there holds the inequality

$$u + v > u_+ . \qquad (14)$$

Let us introduce the functions $p_m = \partial p/\partial m$ and $v_m = \partial v/\partial m$ which make it possible to analyze the change of position of the integral curves with changing

m. From Eqs. (6), (7) we find the equations for p_m and v_m

$$\frac{dp_m}{du} = 1 - \frac{f}{p} v_m + \frac{vf}{p^2} p_m, \qquad (15)$$

$$\frac{dv_m}{du} = \frac{(u + v - u_+)(p - mp_m)}{\lambda p^2} + \frac{m}{\lambda p} v_m.$$

At the point $u = u_+$, at which $p_m = v_m = 0$, there holds

$$\frac{dp_m}{du} > 0, \qquad \frac{dv_m}{du} > 0. \qquad (16)$$

From (16) it follows that in the neighborhood of the point $u = u_+$, the functions p_m and v_m take on negative values. From (15) it is clear that the sign of the functions p_m, v_m does not change over the whole interval $\varepsilon \le u \le u_+$. In fact, assume that at some point the function p_m passes through zero and the function v_m retains its negative value. At that point the curve $p_m(u)$ cannot have a positive derivative. But from (15) it follows that at that point there should hold

$$\frac{dp_m}{du} = 1 - \frac{fv_m}{p} > 0.$$

Assume now that the function v_m passes through zero at a point at which $p_m < 0$. The curve $v_m(u)$ cannot have a positive slope at that point, but from equation (15) and inequality (14) there follows that at that point

$$\frac{dv_m}{du} = \frac{(u + v - u_+)(p - mp_m)}{\lambda p^2} > 0.$$

In an analogous way it can be proved that the case of simultaneous change of sign of p_m and v_m is also impossible.

Thus, the functions p_m and v_m cannot take on positive values over the whole interval $\varepsilon \le u < u_+$. This means that, with m increasing upwards from m = 0, the ordinates of the integral curves $p(u, m)$ increase from the values given by (12) at all points of this interval (including the point $u = \varepsilon$). Consequently, with increasing m the ordinates of $p(\varepsilon - 0, m)$ and $p(\varepsilon + 0, m)$ converge monotonically; and at some finite $m = m^*$, at which $p(\varepsilon - 0, m^*) = p(\varepsilon + 0, m^*)$, there exists a continuous integral curve $p(u, m^*)$ which is the solution of problem (6)–(8). The function $v(u, m^*)$ is then defined by (13).

The case $\lambda = 0$ ($D = 0$) requires separate treatment. In this case we obtain from Eqs. (1), (5) the problem

$$\frac{dp}{du} = m - \frac{f}{m} - \frac{(u_+ - u)f}{p}, \qquad p(u_-) = p(u_+) = 0 \qquad (17)$$

instead of (6), (7).

To prove the existence and uniqueness of the solution of problem (17), consider the auxiliary problem

$$\frac{dp_1}{du} = m - \frac{(u_+ - u)f(u)}{p_1}, \qquad p_1(u_-) = p_1(u_+) = 0. \qquad (18)$$

This problem always has a unique solution [1, 4]. Let us denote by $m°$ the corresponding eigenvalue of (18).

From Eq. (17) it follows that any integral curve of this equation which passes through the point $p(u_+) = 0$ is such that $p(\varepsilon, m) > 0$ for $m > 0$. At $m = m°$ this integral curve lies above the corresponding integral curve of (18). Consequently $p(\varepsilon + 0, m°) > p(\varepsilon - 0, m°)$. At the same time the integral curve of (17) which passes through the point $p(u_-) = 0$ and which is defined by (9) coincides with the solution of (18) over the interval $u \le u \le \varepsilon$. Considering the behavior of the integral curves of (17) when m increases upwards from the value $m = m°$, we conclude that problem (17) has a unique solution.

Note that the above proof is valid also in the case of more general dependence of the rate of the chemical reaction on concentration and temperature, the only condition being that

$$\left.\frac{\partial F}{\partial u}\right|_{u=u_+} = 0, \qquad \frac{\partial F}{\partial v} > 0 \quad \text{for } \varepsilon < u \leqslant u_+.$$

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